Remarks on the behavior of characteristic functions at the origin

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Summary. Some results are obtained that deal with the remainder term of limited expansion in powers of characteristic functions.

1. Main results.

Let $f(t)$ be the characteristic function of a distribution function $F(x)$. Boas [1] investigated the behavior of $f(t)$ in a neighbourhood of the origin, and found a condition on $F(x)$ in order that $\int_0^1 |f(t)-1|\, t^{-p-1} dt < \infty$ with $0 \leqq p < 1$ as well as one in order that $\limsup_{t\to 0+} t^{-p}|f(t)-1| < \infty$ with $0 < p < 1$. The purpose of this paper is to generalize Boas' results by establishing the following theorems.

Theorem 1 *Let $p > 0$ that is not an integer, and let m denote the maximum integer less than p.* (i) *In order that there exist some constants $a_0, a_1, \ldots, a_m$ for which*

$$f(t) - \sum_{j=0}^{m} a_j t^j = o(|t|^p) \quad as \quad t \to 0, \tag{1.1}$$

it is necessary and sufficient that

$$1 - F(x) + F(-x) = o(x^{-p}) \quad as \quad x \to \infty. \tag{1.2}$$

In the case, $a_j = f^{(j)}(0)/j!$ for $j = 0, 1, \ldots, m$. The assertion remains valid if o is replaced by O. (ii) *In order that there exist some constants $a_0, a_1, \ldots, a_m$ for which*

$$\int_{|t|\leqq 1} \Big| f(t) - \sum_{j=0}^{m} a_j t^j \Big| |t|^{-p-1} dt < \infty, \tag{1.3}$$

it is necessary and sufficient that

$$\int_{-\infty}^{\infty}|x|^p dF(x) < \infty. \tag{1.4}$$

In the case, $a_j = f^{(j)}(0)/j!$ *for* $j = 0, 1, \ldots, m$.

Assertion (i) of the theorem has been established by Wolfe [6].

Theorem 2 *Let* $2k$ *be a nonnegative even integer.* (i) *In order that there exist some constants* $a_0, a_1, \ldots, a_{2k}$ *for which*

$$f(t) - \sum_{j=0}^{2k} a_j t^j = o(t^{2k}) \quad as \quad t \to 0, \tag{1.5}$$

it is necessary and sufficient that

$$\int_{-\infty}^{\infty} x^{2k} dF(x) < \infty. \tag{1.6}$$

In the case, $a_j = f^{(j)}(0)/j!$ *for* $j = 0, 1, \ldots, 2k$. (ii) *In order that there exist some constants* $a_0, a_1, \ldots, a_{2k}$ *for which*

$$\int_{|t| \leqq 1} \Big| f(t) - \sum_{j=0}^{2k} a_j t^j \Big| |t|^{-2k-1} dt < \infty, \tag{1.7}$$

it is necessary and sufficient that

$$\int_{|x|>1} x^{2k} \log |x| dF(x) < \infty. \tag{1.8}$$

In the case, $a_j = f^{(j)}(0)/j!$ *for* $j = 0, 1, \ldots, 2k$.

Assertion (i) of the theorem includes the result in Cramèr [2] p. 90, which gives a necessary and sufficient condition on $F(x)$ for the existence of even derivatives of $f(t)$ at the origin.

Theorem 3 *Let* m *be a positive odd integer.* (i) *In order that there exist some constants* $a_0, a_1, \ldots, a_m$ *for which*

$$f(t) - \sum_{j=0}^{m} a_j t^j = o(|t|^m) \quad as \quad t \to 0, \tag{1.9}$$

it is necessary and sufficient that

$$1 - F(x) + F(-x) = o(x^{-m}) \quad as \quad x \to \infty \tag{1.10}$$

and further that there exists a constant γ for which

$$\int_{-x}^{x} u^m dF(u) - \gamma = o(1) \quad as \quad x \to \infty. \tag{1.11}$$

In the case, $f^{(m)}(0) = i^m \gamma$ and $a_j = f^{(j)}(0)/j!$ for $j = 0, 1, \ldots, m$. (ii) *In order that there exist some constants $a_0, a_1, \ldots, a_m$ for which*

$$\int_{|t| \leqq 1} \left| f(t) - \sum_{j=0}^{m} a_j t^j \right| |t|^{-m-1} dt < \infty, \tag{1.12}$$

it is necessary and sufficient that

$$\int_{-\infty}^{\infty} |x|^m dF(x) < \infty \tag{1.13}$$

and further that

$$\int_1^{\infty} \left| \int_{|u|>x} u^m dF(u) \right| d \log x < \infty. \tag{1.14}$$

In the case, $a_j = f^{(j)}(0)/j!$ for $j = 0, 1, \ldots, m$.

Assertion (i) of the theorem includes the theorem of Pitman [5], which gives a necessary and sufficient condition on $F(x)$ for the existence of odd derivatives of $f(t)$ at the origin.

REMARKS. (i) The function $T_*(x) = 1 - F(x) - F(-x)$, $x > 0$, is called the *tail difference* of $F(x)$. As is easily seen, $\int_{-x}^{x} u^m dF(u) = -\int_0^x u^m dT_*(u)$. If $F(x)$ satisfies (1.13) and if

$$\int_1^{\infty} x^m \log x \, |dT_*(x)| < \infty, \tag{1.15}$$

then $F(x)$ satisfies (1.14) and

$$\lim_{N \to \infty} \int_{1 < |x| \leqq N} x^m \log |x| \, dF(x) \quad \text{exists finitely.} \tag{1.16}$$

Conversely, (1.15) is derived from (1.14) or (1.16) when $T_*(x)$ is nonincreasing over an unbounded interval of the positive axis or, equivalently, $F(-I) \leqq F(I)$ for every interval I contained in the unbounded interval. Hence, on the assumption that $F(x)$ has the moment of order m and that $T_*(x)$ is monotone over an unbounded interval of the positive axis, conditions (1.14)–(1.16) are equivalent to each other.

(ii) When $F(x)$ is concentrated on the positive axis, a necessary and sufficient condition for (1.9) is that

$$\int_0^\infty x^m dF(x) < \infty, \tag{1.17}$$

while a necessary and sufficient condition for (1.12) is that

$$\int_1^\infty x^m \log x \, dF(x) < \infty. \tag{1.18}$$

Actually, in the case, condition (1.17) is equivalent to (1.11) and implies (1.10), while condition (1.18) is equivalent to (1.14) and implies (1.13).

(iii) When $F(x)$ is symmetric with respect to the origin, conditions (1.11) and (1.14) may be omitted since in the case those conditions are trivially satisfied.

Condition (1.13) implies (1.10) and (1.11). Conversely, (1.13) is implied by (1.11) if m is an even integer. The condition that $\int_{|x|>1} |x|^m \log |x| dF(x) < \infty$ implies (1.13) and (1.14). Conversely, this condition is implied by (1.14) if m is an even integer. Thus Theorem 3 is still true for nonnegative even integers m and is reduced to Theorem 2 in the case.

If o were replaced by O in the statements of assertion (i) of Theorem 3 with an even or odd integer m, then that assertion is no longer valid. For the validity of that assertion with O in place of o, its statements should be modified as in follows:

Theorem 4 *Let m be a positive integer. In order that there exist some constants $a_0, a_1, \ldots, a_{m-1}$ for which*

$$f(t) - \sum_{j=0}^{m-1} a_j t^j = O(|t|^m) \quad as \quad t \to 0, \tag{1.19}$$

it is necessary and sufficient that

$$1 - F(x) + F(-x) = O(x^{-m}) \quad as \quad x \to \infty \tag{1.20}$$

and further that

$$\int_{-x}^{x} u^m dF(u) = O(1) \quad as \quad x \to \infty. \tag{1.21}$$

In the case, $a_j = f^{(j)}(0)/j!$ *for* $j = 0, 1, \ldots, m-1$.

In consequence of the theorem, condition (1.19) is equivalent to (1.9) when m is an even integer or $F(x)$ is concentrated on the positive axis.

Notice that $\Re f(t)$ is the characteristic function of the symmetric distribution function $(F(x)+1-F(-x-0))/2$. Hence the following corollary is obtained as an immediate consequence of Theorems 1, 3, and 4.

Corollary *Let* $p > 0$ *that is not an even integer, and let* $2k$ *denote the maximum even integer less than* p. (i) *In order that there exist some constants* $a_0, a_2, \ldots, a_{2k}$ *for which*

$$\Re f(t) - \sum_{j=0}^{k} a_{2j} t^{2j} = o(t^p) \quad as \quad t \to 0+, \tag{1.22}$$

it is necessary and sufficient that

$$1 - F(x) + F(-x) = o(x^{-p}) \quad as \quad x \to \infty. \tag{1.23}$$

In the case, $a_{2j} = f^{(2j)}(0)/(2j)!$ *for* $j = 0, 1, \ldots, k$. *The assertion remains valid if* o *is replaced by* O. (ii) *In order that there exist some constants* $a_0, a_2, \ldots, a_{2k}$ *for which*

$$\int_0^1 \Big| \Re f(t) - \sum_{j=0}^{k} a_{2j} t^{2j} \Big| t^{-p-1} dt < \infty, \tag{1.24}$$

it is necessary and sufficient that

$$\int_{-\infty}^{\infty} |x|^p dF(x) < \infty. \tag{1.25}$$

In the case, $a_{2j} = f^{(2j)}(0)/(2j)!$ *for* $j = 0, 1, \ldots, k$.

2. Some lemmas.

As is well known, if $F(x)$ has the moment of integral order $n > 0$, then $f(t)$ is n times differentiable at every point,

$$\lim_{t\to 0}\frac{1}{t^n}\left|f(t)-\sum_{j=0}^{n}\frac{f^{j}(0)}{j!}t^j\right| = 0, \tag{2.1}$$

and

$$f(t)-\sum_{j=0}^{n}\frac{f^{(j)}(0)}{j!}t^j = \frac{1}{\Gamma(n)}\int_0^t (t-u)^{n-1}\big(f^{(n)}(u)-f^{(n)}(0)\big)\,du \tag{2.2}$$

for all t. When n is an even integer, the latter equation yields that

$$\left|\Re f(t)-\sum_{j=0}^{n/2}\frac{f^{(2j)}(0)}{(2j)!}t^{2j}\right| = \frac{1}{\Gamma(n)}\int_0^t (t-u)^{n-1}\left|\Re f^{(n)}(u)-f^{(n)}(0)\right|du, \tag{2.3}$$

because $\Re f^{(n)}(u)-f^{(n)}(0)$ is a function of constant sign.

On the other hand, the following inequalities hold: for all $t > 0$,

$$1-\Re f(t) \geqq \frac{t^2}{3}\int_{|x|\leqq 1/t} x^2 dF(x), \tag{2.4}$$

$$\frac{1}{t}\int_0^t \big(1-\Re f(u)\big)\,du \geqq \frac{1}{7}\int_{|x|>1/t} dF(x), \tag{2.5}$$

$$\frac{1}{t}\int_0^t \big(1-\Re f(u)\big)\,du \geqq \frac{t^2}{9}\int_{|x|\leqq 1/t} x^2 dF(x). \tag{2.6}$$

For the proof of the first two inequalities, refer to Loève [4] p. 196 or Kawata [3] p. 101. The last inequality is an immediate consequence of the first. In addition to these inequalities, we observe here that

$$\frac{1}{\alpha}\int_0^\infty \big(1-\Re f(u)\big)\,d\Phi(u) \geqq \frac{1}{7}\int_{|x|>0}\left|\Phi(1/|x|)\right|dF(x) \tag{2.7}$$

for any nondecreasing absolutely continuous function $\Phi(t)$ defined on the interval $(0,\infty)$ with $\Phi(\infty)=0$ for which $t^{\alpha}\Phi'(t)$ is nonincreasing for some $\alpha>0$. In fact, it follows at once from (2.5) that

$$\int_0^\infty t^{-1}d\Phi(t)\int_0^t \bigl(1-\Re f(u)\bigr)\,du \geqq \frac{1}{7}\int_0^\infty d\Phi(t)\int\limits_{|x|>1/t} dF(x). \tag{2.8}$$

Exchange the order of integrals in each of the iterated integrals, and we have

$$\int_0^\infty \bigl(1-\Re f(u)\bigr)\,du\int_u^\infty t^{-1}d\Phi(t) \geqq \frac{1}{7}\int\limits_{|x|>0} dF(x)\int_{1/|x|}^\infty d\Phi(t). \tag{2.9}$$

The right hand side equals that of (2.7), while the inner integral of the iterated integral on the left hand side is

$$= \int_u^\infty t^{\alpha}\Phi'(t)\,t^{-1-\alpha}dt \leqq u^{\alpha}\Phi'(u)\int_u^\infty t^{-1-\alpha}dt = \Phi'(u)/\alpha. \tag{2.10}$$

Thereby (2.7) is established. Inequality (2.7) leads to Lemma 2 later.

Lemma 1 *Let $2k$ be a positive even integer. If*

$$\liminf_{t\to 0+}\frac{1}{t^{2k}}\Bigl|\Re f(t)-\sum_{j=0}^{k-1}a_{2j}t^{2j}\Bigr| < \infty \tag{2.11}$$

for some constants $a_0, a_2, \ldots, a_{2k-2}$, then $F(x)$ has the moment of order $2k$.

Proof. We prove the lemma by the induction. When $k=1$, the truth of the lemma follows from (2.4). Now, on the assumption that the lemma is valid for some positive integer k, we show that $F(x)$ has the moment of order $2k+2$ if

$$\liminf_{t\to 0+}\frac{1}{t^{2k+2}}\Bigl|\Re f(t)-\sum_{j=0}^{k}a_{2j}t^{2j}\Bigr| < \infty. \tag{2.12}$$

Since (2.12) implies

$$\liminf_{t\to 0+}\frac{1}{t^{2k}}\Bigl|\Re f(t)-\sum_{j=0}^{k}a_{2j}t^{2j}\Bigr| = 0, \tag{2.13}$$

it follows that

$$\liminf_{t\to 0+} \frac{1}{t^{2k}} \Big| \Re f(t) - \sum_{j=0}^{k-1} a_{2j} t^{2j} \Big| \leqq |a_{2k}|. \tag{2.14}$$

Therefore, by the assumption, $F(x)$ has the moment of order $2k$. Besides, we easily see by virtue of (2.13) and (2.1) that $a_{2j} = f^{(2j)}(0)/(2j)!$ for $j = 0, 1, \ldots, k$. Hence equation (2.3) ensures that (2.12) leads to

$$\liminf_{t\to 0+} \frac{1}{t^{2k+2}} \int_0^t (t-u)^{2k-1} \Big| \Re f^{(2k)}(u) - f^{(2k)}(0) \Big| du < \infty, \tag{2.15}$$

in which $t - u \geqq t/2$ for $u \in (0, t/2)$. Therefore

$$\liminf_{t\to 0+} \frac{1}{t^3} \int_0^t \Big| \Re f^{(2k)}(u) - f^{(2k)}(0) \Big| du < \infty, \tag{2.16}$$

which together with (2.6) deduces the existence of the moment of $F(x)$ of order $2k+2$. This completes the proof of the lemma.

Remarks. Similar arguments as in the proof of Lemma 1 deduce that $F(x)$ should be degenerate at the origin if

$$\liminf_{t\to 0+} \frac{1}{t^{2k}} \Big| \Re f(t) - \sum_{j=0}^{k-1} a_{2j} t^{2j} \Big| = 0 \tag{2.17}$$

for some positive even integer $2k$ and for some constants $a_0, a_2, \ldots, a_{2k-2}$.

Lemma 2 (i) *The inequality*

$$\int_0^1 \big(1 - \Re f(u)\big) u^{-1} du \geqq \frac{1}{7} \int_{|x|>1} \log |x| dF(x) \tag{2.18}$$

holds if the integral on the right hand side exists finitely. (ii) *The inequality*

$$\int_0^\infty \big(1 - \Re f(u)\big) u^{-p-1} du \geqq \frac{1}{7} \int_{-\infty}^{\infty} |x|^p dF(x) \tag{2.19}$$

holds for $0 < p < 2$ *if the integral on the right hand side exists finitely.*

Proof. Apply (2.7) with $\Phi(t) = \min\{\log t, 0\}$ and $\alpha = 1$ or $\Phi(t) = -t^{-p}$ and $\alpha = p+1$.

REMARKS. Inequality (2.19) remains true even where $p \geqq 2$. However, $F(x)$ should be degenerate at the origin if the integral on the left hand side of (2.19) is finite for some $p \geqq 2$. As a matter of fact, the finiteness of that integral implies that $\liminf_{t\to 0+} t^{-p}(1-\Re f(t)) = 0$, whereas (2.4) includes the fact that $F(\{0\}) = 1$ if $\liminf_{t\to 0+} t^{-2}(1-\Re f(t)) = 0$.

Let now n be a positive integer, and write

$$\mathfrak{M}_n(\lambda) = \frac{1}{\lambda^n}\int\limits_{|x|\leqq\lambda} |x|^n dF(x) + \int\limits_{|x|>\lambda} dF(x) \tag{2.20}$$

for $\lambda > 0$. We then see that

$$\left|f(t)-1\right| \leqq 2\,\mathfrak{M}_1(1/t) \quad \text{and} \quad \left|f(t)-1-it\int\limits_{|x|\leqq 1/t} x\,dF(x)\right| \leqq 2\,\mathfrak{M}_2(1/t) \tag{2.21}$$

for all $t > 0$, since $|e^{i\theta}-1| \leqq |\theta|$ and $|e^{i\theta}-1-i\theta| \leqq \theta^2/2$ for all real numbers θ bounded by 1. The function $\mathfrak{M}_n(\lambda)$ admits the representation

$$\mathfrak{M}_n(\lambda) = \frac{1}{\lambda^n}\int_0^\lambda T(x)\,dx^n \tag{2.22}$$

by means of integration by parts. Here $T(x) = 1-F(x)+F(-x-0)$ for $x \geqq 0$, the *tail sum* of $F(x)$.

Lemma 3 (i) *Suppose that*

$$1-F(x)+F(-x) = o(x^{-p}) \quad \textit{as} \quad x\to\infty \tag{2.23}$$

for some $0 < p \leqq 1$. *If* $0 < p < 1$, *then*

$$f(t)-1 = o(t^p) \quad \textit{as} \quad t\to 0+. \tag{2.24}$$

If $p = 1$, *then*

$$f(t)-1-it\int\limits_{|x|\leqq 1/t} x\,dF(x) = o(t) \quad \textit{as} \quad t\to 0+. \tag{2.25}$$

The assertion remains true if o *is replaced by* O. (ii) *Suppose that*

$$\int_{|x|>1} \max\{\log|x|, |x|^p\} dF(x) < \infty \tag{2.26}$$

for some $0 \leqq p \leqq 1$. *If* $0 \leqq p < 1$, *then*

$$\int_0^1 \left| f(t) - 1 \right| t^{-p-1} dt < \infty. \tag{2.27}$$

If $p = 1$, *then*

$$\int_0^1 \left| f(t) - 1 - it \int_{|x| \leqq 1/t} x dF(x) \right| t^{-2} dt < \infty. \tag{2.28}$$

Proof. Throughout the proof, let n be an integer such that $n > p$. (i) Write $Q(x) = x^p T(x)$ for $x > 0$. Given $N > 0$, we have by virtue of (2.22) that

$$\mathfrak{M}_n(1/t) = t^n \int_0^N T(x) dx^n + t^n \int_N^{1/t} Q(x) x^{-p} dx^n \tag{2.29}$$

for all $t < 1/N$. Consequently,

$$\begin{aligned} \mathfrak{M}_n(1/t)\, t^{-p} &\leqq N^n t^{n-p} + \sup_{x \geqq N} Q(x)\, t^{n-p} \int_0^{1/t} x^{-p} dx^n \\ &= N^n t^{n-p} + \frac{n}{n-p} \sup_{x \geqq N} Q(x). \end{aligned} \tag{2.30}$$

Letting $t \to 0+$ and then $N \to \infty$, we obtain

$$\limsup_{t \to 0+} \mathfrak{M}_n(1/t)\, t^{-p} \leqq \frac{n}{n-p} \limsup_{x \to \infty} x^p T(x). \tag{2.31}$$

Hence (i) follows from (2.21). (ii) We have

$$\begin{aligned} &\int_0^1 \mathfrak{M}_n(1/t)\, t^{-p-1} dt \\ &= \int_0^1 t^{n-p-1} dt \int_{|x| \leqq 1/t} |x|^n dF(x) + \int_0^1 t^{-p-1} dt \int_{|x| > 1/t} dF(x). \end{aligned} \tag{2.32}$$

The first summand of (2.32) is

$$= \int_0^1 t^{n-p-1}dt \int\limits_{|x|\leqq 1} |x|^n dF(x) + \int\limits_{|x|>1} |x|^n dF(x) \int_0^{1/|x|} t^{n-p-1}dt$$

$$= \frac{1}{n-p}\int\limits_{|x|\leqq 1} |x|^n dF(x) + \frac{1}{n-p}\int\limits_{|x|>1} |x|^p dF(x), \tag{2.33}$$

while the second is

$$= \int\limits_{|x|>1} dF(x) \int_{1/|x|}^1 t^{-p-1}dt = \frac{1}{p}\int\limits_{|x|>1} (|x|^p - 1)\, dF(x) \tag{2.34}$$

when $p>0$. Consequently,

$$\int_0^1 \mathfrak{M}_n(1/t)\, t^{-p-1}dt \leqq \frac{1}{n-p} + \frac{n}{p\,(n-p)}\int\limits_{|x|>1} |x|^p dF(x). \tag{2.35}$$

When $p=0$, the last summand on the right hand side should be replaced with $\int_{|x|>1} \log|x|\, dF(x)$. Hence (ii) follows from (2.21). This completes the proof of the lemma.

Lemma 4 *Suppose that $F(x)$ has the moment of integral order $m \geqq 0$, and write*

$$R(t) = f(t) - \sum_{j=0}^{m} \frac{f^{(j)}(0)}{j!} t^j - \frac{(it)^{m+1}}{(m+1)!}\int\limits_{|x|\leqq 1/t} x^{m+1} dF(x) \tag{2.36}$$

for $t>0$. (i) *If*

$$1 - F(x) + F(-x) = o(x^{-m-1}) \quad as \quad x \to \infty, \tag{2.37}$$

then

$$R(t) = o(t^{m+1}) \quad as \quad t \to 0+. \tag{2.38}$$

The assertion remains true if o is replaced by O. (ii) *If*

$$\int_{-\infty}^{\infty} |x|^{m+1} dF(x) < \infty, \tag{2.39}$$

then

$$\int_0^1 |R(t)|\, t^{-m-2} dt < \infty. \tag{2.40}$$

Proof. The special case of the lemma with $m=0$ is included in Lemma 3. Let $m \geqq 1$. It can be shown that

$$\frac{1}{\Gamma(m)} \int_0^t (t-u)^{m-1} \Big(u \int_{|x| \leqq 1/u} x^{m+1} dF(x) \Big)\, du$$

$$= \frac{t^{m+1}}{(m+1)!} \int_{|x| \leqq 1/t} x^{m+1} dF(x) + \frac{1}{(m+1)!} \int_{|x|>1/t} (\operatorname{sgn} x)^{m+1}\, dF(x). \tag{2.41}$$

If we write

$$A(u) = f^{(m)}(u) - f^{(m)}(0) - i^{m+1} u \int_{|x| \leqq 1/u} x^{m+1} dF(x), \tag{2.42}$$

then we have by (2.2) and (2.41) that

$$\Big| \frac{1}{\Gamma(m)} \int_0^t (t-u)^{m-1} A(u)\, du - R(t) \Big| \leqq \frac{1}{(m+1)!} \int_{|x|>1/t} dF(x) \tag{2.43}$$

for all $t>0$. (i) If $F(x)$ satisfies (2.37), then it follows from Lemma 3 (i) that $A(u)=o(u)$ as $u \to 0+$ and hence

$$\int_0^t (t-u)^{m-1} \big|A(u)\big|\, du \leqq \frac{1}{m} \max_{0 \leqq u \leqq t} \big|A(u)\big|\, t^m = o(t^{m+1}) \quad \text{as} \quad t \to 0+. \tag{2.44}$$

The truth of (2.38) is thereby derived from (2.43). The arguments above with O in place of o remain true. (ii) If $F(x)$ satisfies (2.39), then it follows from Lemma 3 (ii) that $\int_0^1 |A(u)| u^{-2} du < \infty$ and hence

$$\int_0^1 t^{-m-2} dt \int_0^t (t-u)^{m-1} \big|A(u)\big|\, du = \int_0^1 \big|A(u)\big|\, du \int_u^1 t^{-m-2} (t-u)^{m-1} dt$$

$$= \int_0^1 \big|A(u)\big| u^{-2} du \int_u^1 s(1-s)^{m-1} ds \leqq \frac{1}{m(m+1)} \int_0^1 \big|A(u)\big| u^{-2} du < \infty. \tag{2.45}$$

The truth of (2.40) is thereby derived from (2.43). This completes the proof of the lemma.

3. Proofs of Theorems.

Proof of Theorem 1.

Let $p > 0$ that is not an integer, and let $2k$ denote the maximum even integer less than p.

(i) Suppose (1.1). Then, since

$$\Re f(t) - \sum_{j=0}^{k} a_{2j} t^{2j} = o(t^p) \quad \text{as} \quad t \to 0+, \tag{3.1}$$

it follows at once from Lemma 1 that $F(x)$ has the moment of order $2k$ and from (2.1) that $a_{2j} = f^{(2j)}(0)/(2j)!$ for $j = 0, 1, \ldots, k$. Recall (2.3), and we obtain

$$\int_0^t (t-u)^{2k-1} \left| \Re f^{(2k)}(u) - f^{(2k)}(0) \right| du = o(t^p) \quad \text{as} \quad t \to 0+ \tag{3.2}$$

when $k > 0$. The integral is

$$\geqq \left(\frac{t}{2}\right)^{2k-1} \int_0^{t/2} \left| \Re f^{(2k)}(u) - f^{(2k)}(0) \right| du. \tag{3.3}$$

Consequently,

$$\frac{1}{t} \int_0^t \left| \Re f^{(2k)}(u) - f^{(2k)}(0) \right| du = o(t^{p-2k}) \quad \text{as} \quad t \to 0+. \tag{3.4}$$

When $k = 0$, this fact directly follows from (3.1). Write $F_{2k}(x) = \int_{-\infty}^{x} u^{2k} dF(u)$, and we have by (2.5) that

$$\int\limits_{|x| > 1/t} dF_{2k}(x) = o(t^{p-2k}) \quad \text{as} \quad t \to 0+, \tag{3.5}$$

which implies (1.2). Thereby (1.2) is deduced from (1.1).

Suppose conversely (1.2). It is easily seen that $F(x)$ has the moment of order m and that $\int_{|u|>x} |u|^m dF(u) = o(x^{-(p-m)})$ as $x \to \infty$. Apply Lemma 3 (i), and we have

$$f^{(m)}(t) - f^{(m)}(0) = o(|t|^{p-m}) \quad \text{as} \quad t \to 0. \tag{3.6}$$

When $m > 0$, it follows from (2.2) that

$$\begin{aligned}
&\Big|f(t) - \sum_{j=0}^{m} \frac{f^{(j)}(0)}{j!} t^j\Big| \\
&\leqq \frac{1}{\Gamma(m)} \int_0^t (t-u)^{m-1} \big|f^{(m)}(u) - f^{(m)}(0)\big|\, du \\
&\leqq \frac{1}{\Gamma(m+1)} \max_{0 \leqq u \leqq t} \big|f^{(m)}(u) - f^{(m)}(0)\big|\, t^m, \qquad (3.7)
\end{aligned}$$

which implies the truth of (1.1) with $f^{(j)}(0)/j!$ in place of a_j for $j = 0, 1, \dots, m$. Thereby (1.1) is deduced from (1.2). Evidently, constants $a_0, a_1, \dots, a_m$ in (1.1) are uniquely determined. The arguments above remain true if o is replaced by O.

(ii) Suppose (1.3). Then, since

$$\int_0^1 \Big|\Re f(t) - \sum_{j=0}^{k} a_{2j} t^{2j}\Big| t^{-p-1}\, dt < \infty, \qquad (3.8)$$

it follows at once from Lemma 1 that $F(x)$ has the moment of order $2k$ and from (2.1) that $a_{2j} = f^{(2j)}(0)/(2j)!$ for $j = 0, 1, \dots, k$. Recall (2.2), and we obtain

$$\int_0^1 t^{-p-1}\, dt \int_0^t (t-u)^{2k-1} \big|\Re f^{(2k)}(u) - f^{(2k)}(0)\big|\, du < \infty \qquad (3.9)$$

when $k > 0$. The iterated integral is

$$\begin{aligned}
&= \int_0^1 \big|\Re f^{(2k)}(u) - f^{(2k)}(0)\big|\, du \int_u^1 t^{-p-1}(t-u)^{2k-1}\, dt \\
&= \int_0^1 \big|\Re f^{(2k)}(u) - f^{(2k)}(0)\big|\, u^{-(p-2k)-1}\, du \int_u^1 s^{p-2k}(1-s)^{2k-1}\, ds \\
&\geqq C \int_0^{1/2} \big|\Re f^{(2k)}(u) - f^{(2k)}(0)\big|\, u^{-(p-2k)-1}\, du, \qquad (3.10)
\end{aligned}$$

where $C = \int_{1/2}^1 s^{p-2k}(1-s)^{2k-1}\, ds$. Consequently,

$$\int_0^1 \big|\Re f^{(2k)}(u) - f^{(2k)}(0)\big|\, u^{-(p-2k)-1}\, du < \infty. \qquad (3.11)$$

When $k=0$, this fact is equivalent to (3.8). Let $F_{2k}(x)$ be the same as before. It then follows from Lemma 2 (ii) that

$$\int_{|x|>1} |x|^{p-2k}\, dF_{2k}(x) < \infty, \tag{3.12}$$

which implies (1.4). Thereby (1.4) is deduced from (1.3).

Suppose conversely (1.4). Then, since $F(x)$ has the moment of order m and $\int_{-\infty}^{\infty}|x|^{p-m}|x|^m dF(x) < \infty$, applying Lemma 3 (ii) yields that

$$\int_0^1 \left|f^{(m)}(u) - f^{(m)}(0)\right| u^{-(p-m)-1}\, du < \infty. \tag{3.13}$$

When $m>0$, it follows from (2.2) that

$$\begin{aligned}
&\int_0^1 \left| f(t) - \sum_{j=0}^{m} \frac{f^{(j)}(0)}{j!} t^j \right| t^{-p-1} dt \\
&\leqq \frac{1}{\Gamma(m)} \int_0^1 t^{-p-1} dt \int_0^t (t-u)^{m-1} \left| f^{(m)}(u) - f^{(m)}(0)\right| du \\
&= \frac{1}{\Gamma(m)} \int_0^1 \left| f^{(m)}(u) - f^{(m)}(0)\right| du \int_u^1 t^{-p-1}(t-u)^{m-1} dt \\
&\leqq \frac{B(p-m+1, m)}{\Gamma(m)} \int_0^1 \left| f^{(m)}(u) - f^{(m)}(0)\right| u^{-(p-m)-1} du,
\end{aligned} \tag{3.14}$$

which implies the truth of (1.3) with $f^{(j)}(0)/j!$ in place of a_j for $j=0, 1, \ldots, m$. Thereby (1.3) is deduced from (1.4). Evidently, constants $a_0, a_1, \ldots, a_m$ in (1.3) are uniquely determined. We thus have completed the proof of Theorem 1.

Proof of Theorem 2.

Assertion (i) of the theorem is substantially included in Lemma 1. On the other hand, assertion (ii) of the theorem is proved in a similar way as in the proof of the preceding theorem with $p=2k$, a nonnegative even integer. As a matter of course, (3.12) should be replaced by

$$\int_{|x|>1} \log|x|\, dF_{2k}(x) < \infty, \tag{3.15}$$

while (3.13) with $p=2k=m$ should be derived from (3.15).

Proof of Theorem 3.

Throughout the proof, we write $F_{m-1}(x)=\int_{-\infty}^{x}u^{m-1}dF(u)$ when $F(x)$ has the moment of order $m-1$.

(i) Suppose (1.9). It readily follows from Theorem 2 (i) that $F(x)$ has the moment of order $m-1$ and $a_j=f^{(j)}(0)/j!$ for $j=0,1,\dots,m-1$. Hence (1.9) is in turn

$$f(t)-\sum_{j=0}^{m-1}\frac{f^{(j)}(0)}{j!}t^j-a_m t^m=o(|t|^m)\quad\text{as}\quad t\to 0. \tag{3.16}$$

Besides, the same argument as in obtaining (3.5) yields that

$$\int\limits_{|x|>1/t} dF_{m-1}(x)=o(t)\quad\text{as}\quad t\to 0+. \tag{3.17}$$

Evidently, (3.17) implies (1.10). Apply Lemma 4 (i), and we have

$$f(t)-\sum_{j=0}^{m-1}\frac{f^{(j)}(0)}{j!}t^j-\frac{(it)^m}{m!}\int\limits_{|x|\leqq 1/t}x^m dF(x)=o(t^m)\quad\text{as}\quad t\to 0+, \tag{3.18}$$

which together with (3.16) yields that

$$\frac{i^m}{m!}\int\limits_{|x|\leqq 1/t}x^m dF(x)-a_m=o(1)\quad\text{as}\quad t\to 0+. \tag{3.19}$$

Hence (1.11) is true with $\gamma=m!\,a_m/i^m$. Thereby (1.10) and (1.11) are deduced from (1.9).

Suppose conversely (1.10) and (1.11). We then have

$$\int\limits_{|u|>x} dF_{m-1}(u)=o(x^{-1})\quad\text{as}\quad x\to\infty \tag{3.20}$$

and

$$\int\limits_{|x|\leqq 1/t} x\,dF_{m-1}(x)-\gamma=o(1)\quad\text{as}\quad t\to 0+. \tag{3.21}$$

It therefore follows from Lemma 3 (i) that

$$f^{(m-1)}(t) - f^{(m-1)}(0) - i^m \gamma t = o(|t|) \quad \text{as} \quad t \to 0, \tag{3.22}$$

which means that $f^{(m-1)}(t)$ is differentiable at the origin and $f^{(m)}(0) = i^m \gamma$. Hence equation (2.2) deduces the truth of (1.9) with $f^{(j)}(0)/j!$ in place of a_j for $j = 0, 1, \ldots, m$. Thereby (1.9) is deduced from (1.10) and (1.11). Evidently, constants $a_0, a_1, \ldots, a_m$ in (1.9) are uniquely determined.

(ii) Suppose (1.12). It readily follows from Theorem 2 (ii) that $F(x)$ has the moment of order $m-1$ and $a_j = f^{(j)}(0)/j!$ for $j = 0, 1, \ldots, m-1$. Hence (1.12) is in turn

$$\int_{|t| \leqq 1} |f(t) - \sum_{j=0}^{m-1} \frac{f^{(j)}(0)}{j!} t^j - a_m t^m| \, |t|^{-m-1} dt < \infty \tag{3.23}$$

Besides, the same argument as in obtaining (3.12) yields that

$$\int_{|x|>1} |x| \, dF_{m-1}(x) < \infty. \tag{3.24}$$

Evidently, (3.24) implies (1.13). Apply Lemma 4 (ii), and we have

$$\int_0^1 \Big| f(t) - \sum_{j=0}^{m-1} \frac{f^{(j)}(0)}{j!} t^j - \frac{(it)^m}{m!} \int_{|x| \leqq 1/t} x^m dF(x) \Big| t^{-m-1} dt < \infty, \tag{3.25}$$

which together with (3.23) yields that

$$\int_0^1 \Big| \frac{i^m}{m!} \int_{|x| \leqq 1/t} x^m dF(x) - a_m \Big| t^{-1} dt < \infty. \tag{3.26}$$

As is easily seen, $a_m = (i^m/m!) \int_{-\infty}^{\infty} x^m \, dF(x)$. Hence (1.14) is true. Thereby (1.13) and (1.14) are deduced from (1.12).

Suppose conversely (1.13) and (1.14). We then have

$$\int_{-\infty}^{\infty} |x| \, dF_{m-1}(x) < \infty \tag{3.27}$$

and

$$\int_1^\infty \Big| \int_{|x| \leqq 1/t} x\, dF_{m-1}(x) - \gamma \Big| t^{-1} dt < \infty \tag{3.28}$$

with $\gamma = \int_{-\infty}^{\infty} x\, dF_{m-1}(x) = f^{(m)}(0)/i^m$. Hence it follows from Lemma 3 (ii) that

$$\int_0^1 \Big| f^{(m-1)}(t) - f^{(m-1)}(0) - f^{(m)}(0)t \Big| t^{-2} dt < \infty. \tag{3.29}$$

Hence equation (2.2) deduces the truth of (1.12) with $f^{(j)}(0)/j!$ in place of a_j for $j = 0, 1, \ldots, m$. Thus (1.12) is deduced from (1.13) and (1.14). Evidently, constants $a_0, a_1, \ldots, a_m$ in (1.12) are uniquely determined. We thereby have completed the proof of Theorem 3.

Proof of Theorem 4.

The proof is carried out in the same way as in the proof of Theorem 2 or Theorem 3 according as m is an even integer or an odd integer.

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